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FRACTIONAL VORONOVSKAYA ASYMPTOTIC EXPANSIONS BY QUASI-INTERPOLATION NEURAL NETWORK OPERATORS APPLIED TO BROWNIAN MOTION

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Abstract. Here we use quasi-interpolation neural network operators of one hidden layer based on sigmoidal and hyperbolic tangent activation functions. In particular we apply fractional Voronovskaya asymptotic expansions related to the error of approximation of these operators to the unit operator.These are applied to Brownian motion over the two dimensional sphere. So we produce fractional asymptotic expansions for a general expectation of Brownian motion via neural networks. We finish with several interesting specific applications.

1. Introduction

The first author in [1] and [2], see Chapters 2-5, was the first to establish neural network approximation to continuous functions with rates by very specifically defined neural network operators of Cardaliaguet-Euvrard and "Squashing" types, by employing the modulus of continuity of the engaged function or its high order derivative, and producing very tight Jackson type inequalities. He treats there both the univariate and multivariate cases. The defining these operators "bell-shaped" and "squashing" functions are assumed to be compact support. Also the first author inspired by [9], continued his studies on neural networks approximation by introducing and using the proper quasi-interpolation operators of sigmoidal and hyperbolic tangent type which resulted into [5], by treating both the univariate and multivariate cases. He did also the correspoding fractional cases [6],[7].
Here the authors based and inspired by [13], they give Voronovskaya type fractional asymptotic expansions for a general expactation of Brownian motion over the two dimensional sphere, induced by neural networks and at the end they provide many important specialized applications. We are motivated by [8]. For general knowledge about neural networks we recomend [14]-[16]. For recent studies in neural networks we refer to [17]-[26].

2. Basics

We need

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Definition 2.1. Let $\nu > 0$, $n = \lceil \nu \rceil$ ($\lceil \cdot \rceil$ is the ceiling of the number), $f \in AC^n([a,b])$ (space of functions f with $f^{(n-1)} \in AC([a,b])$, absolutely continuous functions). We call left Caputo fractional derivative (see [10], pp. 49-52) the function

$$D_{*a}^{\nu} f(x) = \frac{1}{\Gamma(n-\nu)} \int_a^x (x-t)^{n-\nu-1} f^{(n)}(t)\, dt, \tag{1}$$

$\forall\, x \in [a,b]$, where Γ is the gamma function $\Gamma(\nu) = \int_0^{\infty} e^{-t} t^{\nu-1} dt$, $\nu > 0$. Notice $D_{*a}^{\nu} f \in L_1([a,b])$ and $D_{*a}^{\nu} f$ exists a.e.on $[a,b]$.

We set $D_{*a}^{0} f(x) = f(x)$, $\forall\, x \in [a,b]$.

Definition 2.2. (see also [11], [12]). Let $f \in AC^m([a,b])$, $m = \lceil \alpha \rceil$, $\alpha > 0$. The right Caputo fractional derivative of order $\alpha > 0$ is given by

$$D_{b-}^{\alpha} f(x) = \frac{(-1)^m}{\Gamma(m-\alpha)} \int_x^b (\zeta - x)^{m-\alpha-1} f^{(m)}(\zeta)\, d\zeta, \tag{2}$$

$\forall\, x \in [a,b]$. We set $D_{b-}^{0} f(x) = f(x)$. Notice $D_{b-}^{\alpha} f \in L_1([a,b])$ and $D_{b-}^{\alpha} f$ exists a.e.on $[a,b]$.

Convention 2.1. We assume that

$$D_{*x_0}^{\alpha} f(x) = 0, \text{ for } x < x_0, \tag{3}$$

and

$$D_{x_0-}^{\alpha} f(x) = 0, \text{ for } x > x_0, \tag{4}$$

for all $x, x_0 \in [a,b]$.

See also the related [3], [4].

We consider here the sigmoidal function of logarithmic type

$$\eta(x) = \frac{1}{1+e^{-x}}, \quad x \in \mathbb{R}.$$

It has the properties $\lim_{x \to +\infty} \eta(x) = 1$ and $\lim_{x \to -\infty} \eta(x) = 0$.

This function plays the role of an activation function in the hidden layer of neural networks.

As in [9], we consider

$$K(x) := \frac{1}{2}\left(\eta(x+1) - \eta(x-1)\right), \quad x \in \mathbb{R}. \tag{5}$$

We notice the following properties:

i) $K(x) > 0$, $\forall\, x \in \mathbb{R}$,
ii) $\sum_{k=-\infty}^{\infty} K(x-k) = 1$, $\forall\, x \in \mathbb{R}$,
iii) $\sum_{k=-\infty}^{\infty} K(nx-k) = 1$, $\forall\, x \in \mathbb{R}$; $n \in \mathbb{N}$,
iv) $\int_{-\infty}^{\infty} K(x)\, dx = 1$,
v) K is a density function,
vi) K is even: $K(-x) = K(x)$, $x \geq 0$.

We see that ([9])

$$K(x) = \left(\frac{e^2-1}{2e}\right) \frac{e^{-x}}{(1+e^{-x-1})(1+e^{-x+1})} = \tag{6}$$

$$\left(\frac{e^2-1}{2e^2}\right)\frac{1}{\left(1+e^{x-1}\right)\left(1+e^{-x-1}\right)}.$$

vii) By [9] K is decreasing on $\mathbb{R}_+$, and increasing on $\mathbb{R}_-$.

viii) By [6], Ch. 5, for $n \in \mathbb{N}$, $0 < \beta < 1$, we get

$$\sum_{\left\{\begin{array}{c} k=-\infty \\ : |nx-k| > n^{1-\beta} \end{array}\right.}^{\infty} K(nx-k) < \left(\frac{e^2-1}{2}\right) e^{-n^{(1-\beta)}} = 3.192 e^{-n^{(1-\beta)}}. \tag{7}$$

Denote by $\lfloor \cdot \rfloor$ the integral part of a number. Consider $x \in [a,b] \subset \mathbb{R}$ and $n \in \mathbb{N}$ such that $\lceil na \rceil \leq \lfloor nb \rfloor$.

ix) By [6], Ch. 5, it holds

$$\frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} K(nx-k)} < \frac{1}{K(1)} = 5.250312578, \ \forall\, x \in [a,b]. \tag{8}$$

x) It holds $\lim\limits_{n\to\infty} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} K(nx-k) \neq 1$, for at least some $x \in [a,b]$. See also [6], Ch. 5.

Let $f \in C([a,b])$ and $n \in \mathbb{N}$ such that $\lceil na \rceil \leq \lfloor nb \rfloor$.

We study further (see also [6], Ch. 5) the quasi-interpolation positive linear neural network operator

$$H_n(f,x) := \frac{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right) K(nx-k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} K(nx-k)}, \quad x \in [a,b]. \tag{9}$$

For large enough n we always obtain $\lceil na \rceil \leq \lfloor nb \rfloor$. Also $a \leq \frac{k}{n} \leq b$, iff $\lceil na \rceil \leq k \leq \lfloor nb \rfloor$.

We also consider here the hyperbolic tangent function $\tanh x$, $x \in \mathbb{R}$:

$$\tanh x := \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{e^{2x}-1}{e^{2x}+1}.$$

It has the properties $\tanh 0 = 0$, $-1 < \tanh x < 1$, $\forall\, x \in \mathbb{R}$, and $\tanh(-x) = -\tanh x$. Furthermore $\tanh x \to 1$ as $x \to \infty$, and $\tanh x \to -1$, as $x \to -\infty$, and it is strictly increasing on $\mathbb{R}$. Furthermore it holds $\frac{d}{dx}\tanh x = \frac{1}{\cosh^2 x} > 0$.

This function plays also the role of an activation function in the hidden layer of neural networks.

We further consider

$$M(x) := \frac{1}{4}\left(\tanh(x+1) - \tanh(x-1)\right) > 0, \quad \forall\, x \in \mathbb{R}. \tag{10}$$

We easily see that $M(-x) = M(x)$, that is M is even on $\mathbb{R}$. Obviously M is differentiable, thus continuous.

Here we follow [5]

Proposition 2.2. *$M(x)$ for $x \geq 0$ is strictly decreasing.*

Obviously $M(x)$ is strictly increasing for $x \leq 0$. Also it holds $\lim_{x \to -\infty} M(x) = 0 = \lim_{x \to \infty} M(x)$.

Infact M has the bell shape with horizontal asymptote the x-axis. So the maximum of M is at zero, $M(0) = 0.3809297$.

Theorem 2.3. *We have that* $\sum_{i=-\infty}^{\infty} M(x-i) = 1, \ \forall\, x \in \mathbb{R}$.

Thus

$$\sum_{i=-\infty}^{\infty} M(nx-i) = 1, \quad \forall\, n \in \mathbb{N},\ \forall\, x \in \mathbb{R}.$$

Furthermore we get:
Since M is even it holds $\sum_{i=-\infty}^{\infty} M(i-x) = 1, \ \forall x \in \mathbb{R}$.
Hence $\sum_{i=-\infty}^{\infty} M(i+x) = 1, \ \forall\, x \in \mathbb{R}$, and $\sum_{i=-\infty}^{\infty} M(x+i) = 1, \ \forall\, x \in \mathbb{R}$.

Theorem 2.4. *It holds* $\int_{-\infty}^{\infty} M(x)\, dx = 1$.

So $M(x)$ is a density function on $\mathbb{R}$.

Theorem 2.5. *Let* $0 < \beta < 1$ *and* $n \in \mathbb{N}$. *It holds*

$$\sum_{\left\{\substack{k=-\infty \\ : |nx-k| \geq n^{1-\beta}}\right.}^{\infty} M(nx-k) \leq 2e^4 \cdot e^{-2n^{(1-\beta)}}. \tag{11}$$

Theorem 2.6. *Let* $x \in [a,b] \subset \mathbb{R}$ *and* $n \in \mathbb{N}$ *so that* $\lceil na \rceil \leq \lfloor nb \rfloor$. *It holds*

$$\frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} M(nx-k)} < 4.1488766 = \frac{1}{M(1)}. \tag{12}$$

Also by [5], we obtain

$$\lim_{n \to \infty} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} M(nx-k) \neq 1, \tag{13}$$

for at least some $x \in [a,b]$.

Definition 2.3. Let $f \in C([a,b])$ and $n \in \mathbb{N}$ such that $\lceil na \rceil \leq \lfloor nb \rfloor$.

We further study, as in [5], the quasi-interpolation positive linear neural network operator

$$E_n(f,x) := \frac{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right) M(nx-k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} M(nx-k)}, \quad x \in [a,b]. \tag{14}$$

We find here fractional Voronovskaya type asymptotic expansions for $H_n(f,x)$ and $E_n(f,x)$, $x \in [a,b]$.

For related work on neural networks also see [6], [7].
We need,

Theorem 2.7. *([6], Ch.5) Let $\alpha > 0$, $N \in \mathbb{N}$, $N = \lceil \alpha \rceil$, $f \in AC^N([a,b])$, $0 < \beta < 1$, $x \in [a,b]$, $n \in \mathbb{N}$ large enough. Assume that $\left\| D_{x-}^{\alpha} f \right\|_{\infty,[a,x]}$, $\| D_{*x}^{\alpha} f \|_{\infty,[x,b]} \leq \Theta$, $\Theta > 0$. Then*

$$H_n(f,x) - f(x) = \sum_{j=1}^{N-1} \frac{f^{(j)}(x)}{j!} H_n\left((\cdot - x)^j\right)(x) + o\left(\frac{1}{n^{\beta(\alpha-\varepsilon)}}\right), \tag{15}$$

where $0 < \varepsilon \leq \alpha$.

If $N = 1$, the sum in (15) collapses.

The last (15) implies that

$$n^{\beta(\alpha-\varepsilon)} \left[H_n(f,x) - f(x) - \sum_{j=1}^{N-1} \frac{f^{(j)}(x)}{j!} H_n\left((\cdot - x)^j\right)(x) \right] \to 0, \tag{16}$$

as $n \to \infty$, $0 < \varepsilon \leq \alpha$.

When $N = 1$, or $f^{(j)}(x) = 0$, $j = 1, ..., N-1$, then we derive that

$$n^{\beta(\alpha-\varepsilon)} \left[H_n(f,x) - f(x) \right] \to 0$$

as $n \to \infty$, $0 < \varepsilon \leq \alpha$. Of great interest is the case of $\alpha = \frac{1}{2}$.

Next we also need,

Theorem 2.8. *([6], Ch.5) Let $\alpha > 0$, $N \in \mathbb{N}$, $N = \lceil \alpha \rceil$, $f \in AC^N([a,b])$, $0 < \beta < 1$, $x \in [a,b]$, $n \in \mathbb{N}$ large enough. Assume that $\left\| D_{x-}^{\alpha} f \right\|_{\infty,[a,x]}$, $\| D_{*x}^{\alpha} f \|_{\infty,[x,b]} \leq \Theta$, $\Theta > 0$. Then*

$$E_n(f,x) - f(x) = \sum_{j=1}^{N-1} \frac{f^{(j)}(x)}{j!} E_n\left((\cdot - x)^j\right)(x) + o\left(\frac{1}{n^{\beta(\alpha-\varepsilon)}}\right), \tag{17}$$

where $0 < \varepsilon \leq \alpha$.

If $N = 1$, the sum in (17) collapses.

The last (17) implies that

$$n^{\beta(\alpha-\varepsilon)} \left[E_n(f,x) - f(x) - \sum_{j=1}^{N-1} \frac{f^{(j)}(x)}{j!} E_n\left((\cdot - x)^j\right)(x) \right] \to 0, \tag{18}$$

as $n \to \infty$, $0 < \varepsilon \leq \alpha$.

When $N = 1$, or $f^{(j)}(x) = 0$, $j = 1, ..., N-1$, then we derive that

$$n^{\beta(\alpha-\varepsilon)} \left[E_n(f,x) - f(x) \right] \to 0$$

as $n \to \infty$, $0 < \varepsilon \leq \alpha$. Of great interest is the case of $\alpha = \frac{1}{2}$.

3. About Brownian Motion on 2−Dimensional Sphere

The Brownian motion ([13]) on S^n is a diffusion (Markov) process $W_t, t \geq 0$, on S^n whose transition density is a function $P(t,x,y)$ on $(0,\infty) \times S^n \times S^n$ satisfying

$$\frac{\partial P}{\partial t} = \frac{1}{2} \Delta_n P, \tag{19}$$

and

$$P\left(t,x,y\right)\rightarrow\delta_{x}(y)\quad\text{as}\;\;t\rightarrow0^{+}\tag{20}$$

where Δ_n is the Laplace-Beltrami operator of S^n acting on the x-variables and $\delta_x(y)$ is the delta mass at x, i.e. $P(t,x,y)$ is the **heat kernel** of S^n. The heat kernel exists, it is unique, positive, and smooth in (t,x,y).

We mention,

Proposition 3.1. *([13]) The transition density function of the Brownian motion W_t, $t\geq 0$ on S^2 with radius a it is given by the function*

$$p(t,\varphi)=\frac{1}{4\pi a^{2}}\sum_{n\in\mathbb{N}}(2n+1)\exp\left(-\frac{n(n+1)\sqrt{t}}{a}\right)P_{n}^{0}(\cos\varphi)\tag{21}$$

Theorem 3.2. *([8]) Consider function $g:\mathbb{R}\rightarrow\mathbb{R}$, which is bounded on $[0,\pi]$, i.e. there exists $M>0$ such that $|g(\phi)|\leq M$, for every $\phi\in[0,\pi]$, and Lebesgue measurable on $\mathbb{R}$. Let also $W(t,\phi)$ be the Brownian motion on S^2. Then the expectation*

$$E\left(\left|g(W)\right|\right)(t)=\int_{0}^{\pi}\left|g(\phi)\right|p(t,\phi)d\phi\tag{22}$$

is continuous in t, and

$$E\left(\left|g(W)\right|\right)(t)\leq\pi Mp\left(t_{o},\phi_{0}\right),\tag{23}$$

where

$$p\left(t_{0},\phi_{0}\right)=\max_{(t,\phi)\in[t_{1},t_{2}]\times[0,\pi]}p\left(t,\phi\right),\ \textit{with}\ 0<t_{1}<t_{2}.$$

Here $p(t,\phi)$ is the transition density function of the Brownian motion W_t, $t\geq 0$ on S^2 given by (21).

Proposition 3.3. *([8]) Consider function $g:\mathbb{R}\rightarrow\mathbb{R}$, which is bounded on $[0,\pi]$ and Lebesgue measurable on $\mathbb{R}$. Let also $W(t,\phi)$ be the Brownian motion on S^2. Then the expectation*

$$E\left(\left|g(W)\right|\right)(t)=\int_{0}^{\pi}\left|g(\phi)\right|p(t,\phi)d\phi$$

is differentiable in t, and

$$\frac{\partial E\left(\left|g(W)\right|\right)}{\partial t}=\int_{0}^{\pi}\left|g(\phi)\right|\frac{\partial\left(p(t,\phi)\right)}{\partial t}d\phi,\tag{24}$$

which is continuous in t.

Remark 3.1. We observe that $p(t,\varphi)\in C^{\infty}\left(\mathbb{R}_{+}-\{0\}\right)$ with respect to $t>0$. Acting similarly as in Proposition 3.3, we obtain that $E\left(\left|g(W)\right|\right)^{(j)}:=\frac{\partial^{j}E(|g(W)|)}{\partial t^{j}}$ exist and are continuous for $t>0$ and any $j\in\mathbb{N}$.

4. Main Results

Here we present the following results about Brownian Motion by neural network operators.

Proposition 4.1. *We consider* $E(|g(W)|)(t)$ *as in (22). Let* $\alpha > 0$, $N \in \mathbb{N}$, $N = \lceil \alpha \rceil$, $0 < \beta < 1$, $t \in [t_1, t_2]$, *where* $t_1 > 0, t_1 < t_2, n \in \mathbb{N}$ *large enough. Then*

$$H_n(E(|g(W)|), t) - (E(|g(W)|)(t)) = \sum_{j=1}^{N-1} \frac{E(|g(W)|)^{(j)}(t)}{j!} H_n\left((\cdot - t)^j\right)(t) + o\left(\frac{1}{n^{\beta(\alpha-\varepsilon)}}\right), \tag{25}$$

where $0 < \varepsilon \leq \alpha$.

If $N = 1$, *the sum in (25) collapses.*

The last (25) implies that

$$n^{\beta(\alpha-\varepsilon)} \left[H_n(E(|g(W)|), t) - (E(|g(W)|)(t)) - \sum_{j=1}^{N-1} \frac{E(|g(W)|)^{(j)}(t)}{j!} H_n\left((\cdot - t)^j\right)(t)\right] \to 0, \tag{26}$$

as $n \to \infty$, $0 < \varepsilon \leq \alpha$.

When $N = 1$, *we derive that*

$$n^{\beta(\alpha-\varepsilon)} \left[H_n(E(|g(W)|), t) - (E(|g(W)|)(t))\right] \to 0$$

as $n \to \infty$, $0 < \varepsilon \leq \alpha$. *Of great interest is the case of* $\alpha = \frac{1}{2}$.

Proof. From Theorem 2.7. □

Next we give,

Proposition 4.2. *We consider* $E(|g(W)|)(t)$ *as in (22). Let* $\alpha > 0$, $N \in \mathbb{N}$, $N = \lceil \alpha \rceil$, $t \in [t_1, t_2]$, *where* $t_1 > 0, t_1 < t_2, n \in \mathbb{N}$ *large enough. Then*

$$E_n(E(|g(W)|), t) - (E(|g(W)|)(t)) = \sum_{j=1}^{N-1} \frac{E(|g(W)|)^{(j)}(t)}{j!} E_n\left((\cdot - t)^j\right)(t) + o\left(\frac{1}{n^{\beta(\alpha-\varepsilon)}}\right), \tag{27}$$

where $0 < \varepsilon \leq \alpha$.

If $N = 1$, *the sum in (27) collapses.*

The last (27) implies that

$$n^{\beta(\alpha-\varepsilon)} \left[E_n(E(|g(W)|), t) - (E(|g(W)|)(t)) - \sum_{j=1}^{N-1} \frac{E(|g(W)|)^{(j)}(t)}{j!} E_n\left((\cdot - t)^j\right)(t)\right] \to 0, \tag{28}$$

as $n \to \infty$, $0 < \varepsilon \leq \alpha$.

When $N = 1$, *we derive that*

$$n^{\beta(\alpha-\varepsilon)} \left[E_n(E(|g(W)|), t) - (E(|g(W)|)(t))\right] \to 0$$

as $n \to \infty$, $0 < \varepsilon \leq \alpha$. *Of great interest is the case of* $\alpha = \frac{1}{2}$.

Proof. From Theorem 2.8. ☐

5. Applications

We can apply our main results to the function $g(W) = W$.
Consider the function $g : \mathbb{R} \to \mathbb{R}$, where $g(x) = x$ for every $x \in \mathbb{R}$. Let also $W(t, \phi)$ be the Brownian motion on S^2. Then the expectation

$$E(|W|)(t) = \int_0^{\pi} \phi p(t, \phi) d\phi$$

is continuous in t.
Moreover

Corollary 5.1. *Let $\alpha > 0$, $N \in \mathbb{N}$, $N = \lceil \alpha \rceil$, $0 < \beta < 1$, $t \in [t_1, t_2]$, where $t_1 > 0, t_1 < t_2, n \in \mathbb{N}$ large enough. Then*

$$H_n(E(|W|), t) - (E(|W|)(t)) = \sum_{j=1}^{N-1} \frac{E(|W|)^{(j)}(t)}{j!} H_n\left((\cdot - t)^j\right)(t) + o\left(\frac{1}{n^{\beta(\alpha-\varepsilon)}}\right), \tag{29}$$

where $0 < \varepsilon \leq \alpha$.
If $N = 1$, the sum in (29) collapses.
The last (29) implies that

$$n^{\beta(\alpha-\varepsilon)} \left[H_n(E(|W|), t) - (E(|W|)(t)) - \sum_{j=1}^{N-1} \frac{E(|W|)^{(j)}(t)}{j!} H_n\left((\cdot - t)^j\right)(t) \right] \to 0, \tag{30}$$

as $n \to \infty$, $0 < \varepsilon \leq \alpha$.
When $N = 1$, we derive that

$$n^{\beta(\alpha-\varepsilon)} \left[H_n(E(|W|), t) - (E(|W|)(t)) \right] \to 0$$

as $n \to \infty$, $0 < \varepsilon \leq \alpha$. Of great interest is the case of $\alpha = \frac{1}{2}$.

Proof. From Proposition 4.1. ☐

Next we obtain,

Corollary 5.2. *We consider $N \in \mathbb{N}$, $N = \lceil \alpha \rceil$, $t \in [t_1, t_2]$, where $t_1 > 0, t_1 < t_2, n \in \mathbb{N}$ large enough. Then*

$$E_n(E(|W|), t) - (E(|W|)(t)) = \sum_{j=1}^{N-1} \frac{E(|W|)^{(j)}(t)}{j!} E_n\left((\cdot - t)^j\right)(t) + o\left(\frac{1}{n^{\beta(\alpha-\varepsilon)}}\right), \tag{31}$$

where $0 < \varepsilon \leq \alpha$.
If $N = 1$, the sum in (31) collapses.
The last (31) implies that

$$n^{\beta(\alpha-\varepsilon)} \left[E_n(E(|W|), t) - (E(|W|)(t)) - \sum_{j=1}^{N-1} \frac{E(|W|)^{(j)}(t)}{j!} E_n\left((\cdot - t)^j\right)(t) \right] \to 0, \tag{32}$$

as $n \to \infty$, $0 < \varepsilon \leq \alpha$.

When $N = 1$, *we derive that*

$$n^{\beta(\alpha-\varepsilon)} \left[E_n \left(E \left(|W| \right), t \right) - \left(E \left(|W| \right) (t) \right) \right] \to 0$$

as $n \to \infty$, $0 < \varepsilon \leq \alpha$. *Of great interest is the case of* $\alpha = \frac{1}{2}$.

Proof. From Proposition 4.2. □

For the next corollaries we consider the function $g : \mathbb{R} \to \mathbb{R}$, where $g(x) = cosx$ for every $x \in \mathbb{R}$. Let also $W(t, \phi)$ be the Brownian motion on S^2. Then the expectation

$$E \left(|\cos (W)| \right) (t) = \int_0^\pi |\cos \phi| \, p(t, \phi) d\phi$$

is continuous in t.

It follows,

Corollary 5.3. *Let* $\alpha > 0$, $N \in \mathbb{N}$, $N = \lceil \alpha \rceil$, $0 < \beta < 1$, $t \in [t_1, t_2]$, *where* $t_1 > 0, t_1 < t_2, n \in \mathbb{N}$ *large enough. Then*

$$H_n \left(E \left(|\cos (W)| \right), t \right) - \left(E \left(|\cos (W)| \right) (t) \right) =$$

$$\sum_{j=1}^{N-1} \frac{E \left(|\cos (W)| \right)^{(j)} (t)}{j!} H_n \left((\cdot - t)^j \right) (t) + o \left(\frac{1}{n^{\beta(\alpha-\varepsilon)}} \right), \tag{33}$$

where $0 < \varepsilon \leq \alpha$.

If $N = 1$, *the sum in (33) collapses.*

The last (33) implies that

$$n^{\beta(\alpha-\varepsilon)} \left[H_n \left(E \left(|\cos (W)| \right), t \right) - \left(E \left(|\cos (W)| \right) (t) \right) - \right.$$

$$\left. \sum_{j=1}^{N-1} \frac{E \left(|\cos (W)| \right)^{(j)} (t)}{j!} H_n \left((\cdot - t)^j \right) (t) \right] \to 0, \tag{34}$$

as $n \to \infty$, $0 < \varepsilon \leq \alpha$.

When $N = 1$, *we derive that*

$$n^{\beta(\alpha-\varepsilon)} \left[H_n \left(E \left(|\cos (W)| \right), t \right) - \left(E \left(|\cos (W)| \right) (t) \right) \right] \to 0$$

as $n \to \infty$, $0 < \varepsilon \leq \alpha$. *Of great interest is the case of* $\alpha = \frac{1}{2}$.

Proof. From Proposition 4.1. □

Next we obtain,

Corollary 5.4. *We consider* $N \in \mathbb{N}$, $N = \lceil \alpha \rceil$, $t \in [t_1, t_2]$, *where* $t_1 > 0, t_1 < t_2, n \in \mathbb{N}$ *large enough. Then*

$$E_n \left(E \left(|\cos (W)| \right), t \right) - \left(E \left(|\cos (W)| \right) (t) \right) =$$

$$\sum_{j=1}^{N-1} \frac{E \left(|\cos (W)| \right)^{(j)} (t)}{j!} E_n \left((\cdot - t)^j \right) (t) + o \left(\frac{1}{n^{\beta(\alpha-\varepsilon)}} \right), \tag{35}$$

where $0 < \varepsilon \leq \alpha$.

If $N = 1$, *the sum in (35) collapses.*

The last (35) implies that

$$n^{\beta(\alpha-\varepsilon)}\left[E_n\left(E\left(\left|\cos\left(W\right)\right|\right),t\right)-\left(E\left(\left|\cos\left(W\right)\right|\right)\left(t\right)\right)-\right.$$

$$\left.\sum_{j=1}^{N-1}\frac{E\left(\left|\cos\left(W\right)\right|\right)^{(j)}\left(t\right)}{j!}E_n\left(\left(\cdot-t\right)^j\right)\left(t\right)\right]\to 0, \tag{36}$$

as $n\to\infty$, $0<\varepsilon\leq\alpha$.

When $N=1$, *we derive that*

$$n^{\beta(\alpha-\varepsilon)}\left[E_n\left(E\left(\left|\cos\left(W\right)\right|\right),t\right)-\left(E\left(\left|\cos\left(W\right)\right|\right)\left(t\right)\right)\right]\to 0$$

as $n\to\infty$, $0<\varepsilon\leq\alpha$. *Of great interest is the case of* $\alpha=\frac{1}{2}$.

Proof. From Proposition 4.2. □

Note 5.5. Similar results can be obtain for $g:\mathbb{R}\to\mathbb{R}$, where $g(x)=\sin x, x\in\mathbb{R}$

Let the function $g:\mathbb{R}\to\mathbb{R}$, where $g(x)=\tanh x$ for every $x\in\mathbb{R}$. Let also $W(t,\phi)$ be the Brownian motion on S^2. Then the expectation

$$E\left(\left|\tanh\left(W\right)\right|\right)\left(t\right)=\int_0^{\pi}\left|\tanh(\phi)\right|p(t,\phi)d\phi$$

is continuous in t.
Furthermore,

Corollary 5.6. *Let* $\alpha>0$, $N\in\mathbb{N}$, $N=\lceil\alpha\rceil$, $0<\beta<1$, $t\in[t_1,t_2]$, *where* $t_1>0, t_1<t_2, n\in\mathbb{N}$ *large enough. Then*

$$H_n\left(E\left(\left|\tanh\left(W\right)\right|\right),t\right)-\left(E\left(\left|\tanh\left(W\right)\right|\right)\left(t\right)\right)=$$

$$\sum_{j=1}^{N-1}\frac{E\left(\left|\tanh\left(W\right)\right|\right)^{(j)}\left(t\right)}{j!}H_n\left(\left(\cdot-t\right)^j\right)\left(t\right)+o\left(\frac{1}{n^{\beta(\alpha-\varepsilon)}}\right), \tag{37}$$

where $0<\varepsilon\leq\alpha$.

If $N=1$, *the sum in (37) collapses.*

The last (37) implies that

$$n^{\beta(\alpha-\varepsilon)}\left[H_n\left(E\left(\left|\tanh\left(W\right)\right|\right),t\right)-\left(E\left(\left|\tanh\left(W\right)\right|\right)\left(t\right)\right)-\right.$$

$$\left.\sum_{j=1}^{N-1}\frac{E\left(\left|\tanh\left(W\right)\right|\right)^{(j)}\left(t\right)}{j!}H_n\left(\left(\cdot-t\right)^j\right)\left(t\right)\right]\to 0, \tag{38}$$

as $n\to\infty$, $0<\varepsilon\leq\alpha$.

When $N=1$, *we derive that*

$$n^{\beta(\alpha-\varepsilon)}\left[H_n\left(E\left(\left|\tanh\left(W\right)\right|\right),t\right)-\left(E\left(\left|\tanh\left(W\right)\right|\right)\left(t\right)\right)\right]\to 0$$

as $n\to\infty$, $0<\varepsilon\leq\alpha$. *Of great interest is the case of* $\alpha=\frac{1}{2}$.

Proof. From Proposition 4.1. □

Next we give,

Corollary 5.7. *We consider $N \in \mathbb{N}$, $N = \lceil \alpha \rceil$, $t \in [t_1, t_2]$, where $t_1 > 0, t_1 < t_2, n \in \mathbb{N}$ large enough. Then*

$$E_n\left(E\left(\left|\tanh\left(W\right)\right|\right), t\right) - \left(E\left(\left|\tanh\left(W\right)\right|\right)(t)\right) =$$

$$\sum_{j=1}^{N-1} \frac{E\left(\left|\tanh\left(W\right)\right|\right)^{(j)}(t)}{j!} E_n\left(\left(\cdot - t\right)^j\right)(t) + o\left(\frac{1}{n^{\beta(\alpha-\varepsilon)}}\right), \tag{39}$$

where $0 < \varepsilon \leq \alpha$.

If $N = 1$, the sum in (39) collapses.

The last (39) implies that

$$n^{\beta(\alpha-\varepsilon)} \left[E_n\left(E\left(\left|\tanh\left(W\right)\right|\right), t\right) - \left(E\left(\left|\tanh\left(W\right)\right|\right)(t)\right) - \right.$$

$$\left. \sum_{j=1}^{N-1} \frac{E\left(\left|\tanh\left(W\right)\right|\right)^{(j)}(t)}{j!} E_n\left(\left(\cdot - t\right)^j\right)(t)\right] \to 0, \tag{40}$$

as $n \to \infty$, $0 < \varepsilon \leq \alpha$.

When $N = 1$, we derive that

$$n^{\beta(\alpha-\varepsilon)} \left[E_n\left(E\left(\left|\tanh\left(W\right)\right|\right), t\right) - \left(E\left(\left|\tanh\left(W\right)\right|\right)(t)\right)\right] \to 0$$

as $n \to \infty$, $0 < \varepsilon \leq \alpha$. Of great interest is the case of $\alpha = \frac{1}{2}$.

Proof. From Proposition 4.2. $\square$

For the next application we consider the generalized logistic sigmoid function $g : \mathbb{R} \to \mathbb{R}$, where
$g(x) = \left(1 + e^{-x}\right)^{-\delta}$, and $\delta > 0$, for every $x \in \mathbb{R}$. Let also $W(t, \phi)$ be the Brownian motion on S^2. Then the expectation

$$E\left(\left(1 + e^{-W}\right)^{-\delta}\right)(t) = \int_0^\pi \left(1 + e^{-\phi}\right)^{-\delta} p(t, \phi) d\phi$$

is continuous in t.
It follows,

Corollary 5.8. *Let $\alpha > 0$, $N \in \mathbb{N}$, $N = \lceil \alpha \rceil$, $0 < \beta < 1$, $t \in [t_1, t_2]$, where $t_1 > 0, t_1 < t_2, n \in \mathbb{N}$ large enough. Then*

$$H_n\left(E\left(\left(1 + e^{-W}\right)^{-\delta}\right), t\right) - \left(E\left(\left(1 + e^{-W}\right)^{-\delta}\right)(t)\right) =$$

$$\sum_{j=1}^{N-1} \frac{E\left(\left(1 + e^{-W}\right)^{-\delta}\right)^{(j)}(t)}{j!} H_n\left(\left(\cdot - t\right)^j\right)(t) + o\left(\frac{1}{n^{\beta(\alpha-\varepsilon)}}\right), \tag{41}$$

where $0 < \varepsilon \leq \alpha$.

If $N = 1$, the sum in (41) collapses.

The last (41) implies that

$$n^{\beta(\alpha-\varepsilon)} \cdot$$

$$\left[H_n\left(E\left(\left(1 + e^{-W}\right)^{-\delta}\right), t\right) - \left(E\left(\left(1 + e^{-W}\right)^{-\delta}\right)(t)\right) - \right.$$

$$\left.\sum_{j=1}^{N-1}\frac{E\left(\left(1+e^{-W}\right)^{-\delta}\right)^{(j)}(t)}{j!}H_n\left((\cdot-t)^j\right)(t)\right]\to 0, \tag{42}$$

as $n\to\infty$, $0<\varepsilon\le\alpha$.

When $N=1$, *we derive that*

$$n^{\beta(\alpha-\varepsilon)}\left[H_n\left(E\left(\left(1+e^{-W}\right)^{-\delta}\right),t\right)-\left(E\left(\left(1+e^{-W}\right)^{-\delta}\right)(t)\right)\right]\to 0$$

as $n\to\infty$, $0<\varepsilon\le\alpha$. *Of great interest is the case of* $\alpha=\frac{1}{2}$.

Proof. From Proposition 4.1. □

Next we derive,

Corollary 5.9. *We consider* $N\in\mathbb{N}$, $N=\lceil\alpha\rceil$, $t\in[t_1,t_2]$, *where* $t_1>0, t_1<t_2, n\in\mathbb{N}$ *large enough. Then*

$$E_n\left(E\left(\left(1+e^{-W}\right)^{-\delta}\right),t\right)-\left(E\left(\left(1+e^{-W}\right)^{-\delta}\right)(t)\right)=$$

$$\sum_{j=1}^{N-1}\frac{E\left(\left(1+e^{-W}\right)^{-\delta}\right)^{(j)}(t)}{j!}E_n\left((\cdot-t)^j\right)(t)+o\left(\frac{1}{n^{\beta(\alpha-\varepsilon)}}\right), \tag{43}$$

where $0<\varepsilon\le\alpha$.

If $N=1$, *the sum in (43) collapses.*

The last (43) implies that

$$n^{\beta(\alpha-\varepsilon)}\cdot$$

$$\left[E_n\left(E\left(\left(1+e^{-W}\right)^{-\delta}\right),t\right)-\left(E\left(\left(1+e^{-W}\right)^{-\delta}\right)(t)\right)-\right.$$

$$\left.\sum_{j=1}^{N-1}\frac{E\left(\left(1+e^{-W}\right)^{-\delta}\right)^{(j)}(t)}{j!}E_n\left((\cdot-t)^j\right)(t)\right]\to 0, \tag{44}$$

as $n\to\infty$, $0<\varepsilon\le\alpha$.

When $N=1$, *we derive that*

$$n^{\beta(\alpha-\varepsilon)}\left[E_n\left(E\left(\left(1+e^{-W}\right)^{-\delta}\right),t\right)-\left(E\left(\left(1+e^{-W}\right)^{-\delta}\right)(t)\right)\right]\to 0$$

as $n\to\infty$, $0<\varepsilon\le\alpha$. *Of great interest is the case of* $\alpha=\frac{1}{2}$.

Proof. From Proposition 4.2. □

The Gompertz function $g:\mathbb{R}\to\mathbb{R}$, with $g(x)=e^{\mu e^{-x}}$, $\mu<0$ is a sigmoid function which describes growth as being slowest at the start and end of a given time period. Let $W(t,\phi)$ be the Brownian motion on S^2. Then the expectation

$$E\left(e^{\mu e^{-W}}\right)(t)=\int_0^\pi e^{\mu e^{-\phi}}p(t,\phi)d\phi$$

is continuous in t.

Moreover,

Corollary 5.10. *Let $\alpha > 0$, $N \in \mathbb{N}$, $N = \lceil \alpha \rceil$, $0 < \beta < 1$, $t \in [t_1, t_2]$, where $t_1 > 0, t_1 < t_2, n \in \mathbb{N}$ large enough. Then*

$$H_n\left(E\left(e^{\mu e^{-W}}\right), t\right) - \left(E\left(e^{\mu e^{-W}}\right)(t)\right) =$$

$$\sum_{j=1}^{N-1} \frac{E\left(e^{\mu e^{-W}}\right)^{(j)}(t)}{j!} H_n\left((\cdot - t)^j\right)(t) + o\left(\frac{1}{n^{\beta(\alpha-\varepsilon)}}\right), \tag{45}$$

where $0 < \varepsilon \leq \alpha$.

If $N = 1$, the sum in (45) collapses.

The last (45) implies that

$$n^{\beta(\alpha-\varepsilon)}\left[H_n\left(E\left(e^{\mu e^{-W}}\right), t\right) - \left(E\left(e^{\mu e^{-W}}\right)(t)\right) - \sum_{j=1}^{N-1} \frac{E\left(e^{\mu e^{-W}}\right)^{(j)}(t)}{j!} H_n\left((\cdot - t)^j\right)(t)\right] \to 0, \tag{46}$$

as $n \to \infty$, $0 < \varepsilon \leq \alpha$.

When $N = 1$, we derive that

$$n^{\beta(\alpha-\varepsilon)}\left[H_n\left(E\left(e^{\mu e^{-W}}\right), t\right) - \left(E\left(e^{\mu e^{-W}}\right)(t)\right)\right] \to 0$$

as $n \to \infty$, $0 < \varepsilon \leq \alpha$. Of great interest is the case of $\alpha = \frac{1}{2}$.

Proof. From Proposition 4.1. □

Next we give,

Corollary 5.11. *We consider $N \in \mathbb{N}$, $N = \lceil \alpha \rceil$, $t \in [t_1, t_2]$, where $t_1 > 0, t_1 < t_2, n \in \mathbb{N}$ large enough. Then*

$$E_n\left(E\left(e^{\mu e^{-W}}\right), t\right) - \left(E\left(e^{\mu e^{-W}}\right)(t)\right) =$$

$$\sum_{j=1}^{N-1} \frac{E\left(e^{\mu e^{-W}}\right)^{(j)}(t)}{j!} E_n\left((\cdot - t)^j\right)(t) + o\left(\frac{1}{n^{\beta(\alpha-\varepsilon)}}\right), \tag{47}$$

where $0 < \varepsilon \leq \alpha$.

If $N = 1$, the sum in (47) collapses.

The last (47) implies that

$$n^{\beta(\alpha-\varepsilon)}\left[E_n\left(E\left(e^{\mu e^{-W}}\right), t\right) - \left(E\left(e^{\mu e^{-W}}\right)(t)\right) - \sum_{j=1}^{N-1} \frac{E\left(e^{\mu e^{-W}}\right)^{(j)}(t)}{j!} E_n\left((\cdot - t)^j\right)(t)\right] \to 0, \tag{48}$$

as $n \to \infty$, $0 < \varepsilon \leq \alpha$.

When $N = 1$, we derive that

$$n^{\beta(\alpha-\varepsilon)}\left[E_n\left(E\left(e^{\mu e^{-W}}\right), t\right) - \left(E\left(e^{\mu e^{-W}}\right)(t)\right)\right] \to 0$$

as $n \to \infty$, $0 < \varepsilon \leq \alpha$. *Of great interest is the case of* $\alpha = \frac{1}{2}$.

Proof. From Proposition 4.2. □

Conclusion:

Using mathematical analysis methods we derived Voronovskaya type fractional asymptotic expansions for the general expectation of Brownian motion over the two dimensional sphere. These are realated to neural network approximation operators which are activated by the sigmoidal and hyperbolic tangent functions.

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